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Gauged Gross–Neveu model with overlap fermions*

Ikuo Ichinose and Keiichi Nagao

Institute of Physics, University of Tokyo, Komaba, Tokyo, 153-8902 Japan

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We investigate chiral properties of the overlap lattice fermion by using solvable model in two dimensions, the gauged Gross-Neveu model. In this model, the chiral symmetry is spontaneously broken in the presence of small but finite fermion mass. We calculate the quasi-Nambu-Goldstone(NG) boson mass as a function of the bare fermion mass and two parameters in the overlap formula. We find that the quasi-NG boson mass has desired properties as a result of the extended chiral symmetry found by Lüscher. We also show the PCAC relation is satisfied in desired form. Comparison between the overlap and Wilson lattice fermions is also made.

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I. Introduction

Species doubling is a long standing problem in the lattice fermion formulation. Wilson fermion is the most suitable formulation[1] and it is used in most of the numerical

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studies of lattice gauge theory. However in order to reach the desired continuum limit, fine tuning must be done with respect to the "bare fermion mass" and the Wilson parameter. Recently a very promising formulation of lattice fermion named overlap fermion was proposed by Narayanan and Neuberger[2]. In that formula the Ginsparg and Wilson(GW) relation[3] plays a very important role, and because of that there exists an "extended" (infinitesimal) chiral symmetry. In this paper we shall study or test the overlap fermion by using the gauged Gross-Neveu model in two dimensions. This is a solvable model which has similar chiral properties with QCD_4, i.e., chiral symmetry is spontaneously broken with a small but finite bare fermion mass and pion appears as quasi-Nambu-Goldstone boson. Actually a closely related model was studied on a lattice in order to test properties of the Wilson fermion in the continuum limit[4]. Therefore advantage of the overlap fermion becomes clear by the investigation in this paper.

In order to examine chiral properties of the overlap fermion, we define lattice Gross-Neveu model in two ways.

- In the first one, the interaction terms respect the ordinary chiral symmetry and break the extended chiral symmetry. Then it is not obvious if quasi-massless pions appear (in the continuum limit) with or without any finite tuning of the parameters in the overlap fermion.

- On the other hand, the second one respect the extended chiral symmetry and as a result the interaction terms become nonlocal. Then it is not clear if the Goldstone theorem is applicable in this system.

All these questions are answered in this paper. The first question is important for a formulation of lattice theory with a symmetry which cannot be realized exactly on a lattice like supersymmetry. The second one is important for study of QCD with exact

extended chiral symmerty. Especially study of the second lattice model shows what is an order parameter for the extended chiral symmetry and which field behaves as Nambu-Goldstone boson.

II. The first model

The first model is defined by the following action on lattice with the lattice spacing a,

$$
\begin{aligned}
S &= \frac{N}{2}\sum_{pl}\prod U_\mu(n) + a^2\sum_{n,m}\bar{\psi}(m)D(m,n)\psi(n) + a^2 M_B\sum_n \bar{\psi}\psi(n) \\
&\quad -\frac{a^2}{\sqrt{N}}\sum_n\Big[\phi^i(n)(\bar{\psi}\tau^i\psi)(n) + \phi_5^i(n)(\bar{\psi}\tau^i\gamma_5\psi)(n)\Big] \\
&\quad +\frac{a^2}{2g_v}\sum_n\Big[\phi^i(n)\phi^i(n) + \phi_5^i(n)\phi_5^i(n)\Big],
\end{aligned}
\tag{1}
$$

where $U_\mu(n)$ is U(1) gauge field defined on links, ψ^l_α $(\alpha = 1, ..., N, l = 1, ..., L)$ are fermion fields with flavour index l, and the matrix τ^i $(i = 0, ..., L^2 - 1)$ acting on the flavour index is normalized as

$$
\mathrm{Tr}(\tau^i\tau^k) = \delta_{ik} \tag{2}
$$

and

$$
\tau^0 = \frac{1}{\sqrt{L}}, \quad \{\tau^i, \tau^j\} = d^{ijk}\tau^k, \tag{3}
$$

where d^{ijk}'s are the structure constants of $SU(L)$. Fields ϕ^i and ϕ^i_5 are scalar and pseudo-scalar bosons, respectively. The covariant derivative in Eq.(1) is defined by the overlap formula

$$
D = \frac{1}{a}\Big(1 + X\frac{1}{\sqrt{X^\dagger X}}\Big),
$$

$$
\begin{aligned}
X_{nm} &= \gamma_\mu C_\mu(n,m) + B(n,m), \\
C_\mu &= \frac{1}{2a}\Big[\delta_{m+\mu,n} U_\mu(m) - \delta_{m,n+\mu} U^\dagger_\mu(n)\Big], \\
B(n,m) &= -\frac{M_0}{a} + \frac{r}{2a}\sum_\mu \Big[2\delta_{n,m} - \delta_{m+\mu,n} U_\mu(m) - \delta_{m,n+\mu} U^\dagger_\mu(n)\Big],
\end{aligned} \tag{4}
$$

where r and M_0 are dimensionless nonvanishing free parameters of the overlap lattice fermion formalism[2, 5]. The overlap Dirac operator D does not have the ordinary chiral invariance but satisfies the GW relation instead,

$$
D\gamma_5 + \gamma_5 D = aD\gamma_5 D. \tag{5}
$$

From (1) it is obvious that the systematic $1/N$ expansion is possible and we shall employ it.

The action (1) contains the bare fermion mass M_B which explicitly breaks the chiral symmetry. This bare mass also breaks the following infinitesimal transformation, which was discovered by Lüscher[6] and we call "extended chiral symmetry",

$$
\begin{aligned}
&\psi(n) \to \psi(n) + \tau^k\theta^k\gamma_5\Big\{\delta_{nm} - aD(n,m)\Big\}\psi(m), \\
&\bar\psi(n) \to \bar\psi(n) + \bar\psi(n)\tau^k\theta^k\gamma_5 \\
&\phi^i(n) \to \phi^i(n) + d^{ikj}\theta^k\phi^j_5(n), \\
&\phi^i_5(n) \to \phi^i_5(n) - d^{ikj}\theta^k\phi^j(n),
\end{aligned} \tag{6}
$$

where θ^i is an infinitesimal transformation parameter. It is also verifed that the interaction terms in (1) also explicitly break the extended chiral symmetry.

From the action (1), it is obvious that ϕ^i and ϕ^i_5 are composite fields of the fermions,

$$
\phi^i = \frac{g_v}{\sqrt{N}}\bar\psi\tau^i\psi, \quad \phi^i_5 = \frac{g_v}{\sqrt{N}}\bar\psi\gamma_5\tau^i\psi. \tag{7}
$$

As in the continuum model, we expect that the field ϕ^0 acquires a nonvanishing vacuum expectation value(VEV),

$$
\langle\phi^0\rangle = \sqrt{NL}M_s, \tag{8}
$$

and we define subtracted fields,

$$\varphi^0 = \phi^0 - \sqrt{NL}M_s,$$
$$\varphi^i = \phi^i \ \ (i \neq 0), \ \ \varphi_5^i = \phi_5^i. \tag{9}$$

From the chiral symmetry and (8), one may expect that quasi-Nambu-Goldstone(NG) bosons appear as a result of the spontaneous breaking of the chiral symmetry. They are nothing but φ_5^i. However as we explained above, this expectation can *not* be accepted straightforwardly because of explicit breaking of both the ordinary and extended chiral symmetries. Careful studies are then required.

The VEV M_s is determined by the tadpole cancellation condition of φ^0. In order to perform an explicit calculation of the $1/N$-expansion, we introduce the gauge potential $\lambda_\mu(n)$ in the usual way, i.e., $U(n,\mu) = \exp(\frac{ia}{\sqrt{N}}\lambda_\mu(n))$ and employ the weak-coupling expansion by Kikukawa and Yamada[7].

The effective action of the pion and the gauge boson is obtained by integrating over the quark fields. For details, see Ref.[8]. The pion part is given as follows in the leading-order of $1/N$,

$$S^{(2)}_{eff}[\varphi_5] = \int_k \frac{1}{2}\varphi_5^i(-k)\Gamma^5_{ij}(k^2)\varphi_5^j(k) \tag{10}$$

where

$$\begin{aligned}\Gamma^5_{ij}(k^2) &= \delta_{ij}\Big[\frac{1}{g_v} + \int_k \mathrm{Tr}[\gamma_5\langle\psi(k-p)\bar{\psi}(k-p)\rangle\gamma_5\langle\psi(k)\bar{\psi}(k)\rangle]\Big] \\ &= \delta_{ij}\Big[\epsilon + 2k^2M_0^2A(k^2;M)\Big], \end{aligned} \tag{11}$$
$$M = M_B + M_s. \tag{12}$$

Parameter ϵ in Γ^5_{ij} is proportional to the pion mass and measures the derivation from the limit of the exact chiral symmetry. Practical calculation gives

$$\epsilon = \frac{M_BM_0^2}{M_s}\Big[-\ln(M_0M^2a) + \text{const.}\Big] + O(a). \tag{13}$$

Therefore $\epsilon \propto M_B + O(a)$ and the limit $M_B \to 0$ is considered as the chiral limit. This result is in sharp contrast with the Wilson fermion. There fine tunning of the the "bare mass" $M_{W,B}$ and the Wilson parameter r_W is required in order to reach the chiral limit[4].

It is also straightforward to calculate $A(k^2; M)$ in (11). In the continuum limit,

$$A(k^2; M) \to \frac{1}{4\pi\mu^2} + O(k^2). \tag{14}$$

where $\mu = M_0 M$ and therefore the pion mass is given as $m_\pi^2 = 2\pi M^2 \epsilon$.

There exists a mixing term of the gauge boson λ_μ and the pion φ_5^0,

$$S_{eff}^{(2)}[\lambda_\mu, \varphi_5^0] = -2\sqrt{L} M_0^2 M \int_k \sum \lambda_\mu(-k)\epsilon_{\mu\nu} k_\nu A(k^2; M)\varphi_5^0(k), \tag{15}$$

which is identical with the continuum calculation. This mixing term is related to the discussion of the U(1) problem in QCD_4 and the above result suggests that the correct anomaly appears in the Ward-Takahashi identity of the axial-vector current.

We shall examine the PCAC relation. By changing variables as follows in the path-integral representation of the partition function,

$$\begin{aligned}
&\psi(n) \to \psi(n) + \tau^k \theta^k(n)\gamma_5 \Big\{\delta_{nm} - aD(n,m)\Big\}\psi(m), \\
&\bar{\psi}(n) \to \bar{\psi}(n)\left\{1 + \tau^k\theta^k(n)\gamma_5\right\}, \\
&\phi^i(n) \to \phi^i(n) + d^{ikj}\theta^k\phi_5^j(n), \\
&\phi_5^i(n) \to \phi_5^i(n) - d^{ikj}\theta^k\phi^j(n),
\end{aligned} \tag{16}$$

we obtain the Ward-Takahashi(WT) identity,

$$\begin{aligned}
&\langle \partial_\mu j_{5,\mu}^k(n) - 2M(\bar{\psi}\tau^k\gamma_5\psi)(n) + \frac{2\sqrt{N}}{g_v} M_s \varphi_5^k(n) \\
&\quad + D_A^k(n) - \delta^{k0} N\sqrt{L} a \mathrm{Tr}[\gamma_5 D(n,n)]\rangle = 0,
\end{aligned} \tag{17}$$

where the last term comes from the measure of the path integral, and the explicit form of the current operator $j_{5,\mu}^k$ is obtained by Kikukawa and Yamada[9]. The above WT identity

is expressed in terms of the pions and gauge boson by integrating over the quarks. We obtain the final form of the WT identity *in the continuum limit*,

$$
\begin{aligned}
\partial_\mu j^k_{5,\mu} &= i\delta^{k0}\frac{\sqrt{NL}}{\pi}\sum_{\mu\nu}\epsilon_{\mu\nu}\partial_\nu\lambda_\mu + 2M\epsilon\sqrt{N}\varphi^k_5 \\
&= i\delta^{k0}\frac{\sqrt{NL}}{\pi}\sum_{\mu\nu}\epsilon_{\mu\nu}\partial_\nu\lambda_\mu + \sqrt{\frac{2N}{\pi}}m^2_\pi \times \frac{\varphi^k_5}{\sqrt{2\pi M^2}}. \qquad (18)
\end{aligned}
$$

Then it is obvious that the PCAC relation is satisfied in the overlap fermion formalism.

III. The second model

We shall turn to the second lattice Gross-Neveu model. We modify the interaction terms as follows,

$$
\begin{aligned}
S_2 &= -\frac{N}{2}\sum_{pl}\prod U_\mu(n) + a^2\sum_{n,m}\bar{\psi}(m)D(m,n)\psi(n) - a^2 M_B\sum_n\bar{\psi}\psi(n) \\
&\quad -\frac{a^2}{\sqrt{N}}\sum_n\Big[\phi^i(n)\bar{\psi}(n)\tau^i\Big(\delta_{nm}-\frac{a}{2}D(n,m)\Big)\psi(m) \\
&\quad +\phi^i_5(n)\bar{\psi}(n)\tau^i\gamma_5\Big(\delta_{nm}-\frac{a}{2}D(n,m)\Big)\psi(m)\Big] \\
&\quad +\frac{a^2}{2g_v}\sum_n\Big[\phi^i(n)\phi^i(n)+\phi^i_5(n)\phi^i_5(n)\Big]. \qquad (19)
\end{aligned}
$$

It is straightforward to verify the invariance of the action S_2 under the transformation (6). From the action (19), ϕ^i and ϕ^i_5 are *nonlocal* composite fields of the fermions in the present case,

$$
\phi^i = \frac{g_v}{\sqrt{N}}\bar{\psi}\tau^i\Big(1-\frac{a}{2}D\Big)\psi, \quad \phi^i_5 = \frac{g_v}{\sqrt{N}}\bar{\psi}\tau^i\gamma_5\Big(1-\frac{a}{2}D\Big)\psi. \qquad (20)
$$

As in the previous case we expect that the field ϕ^0 acquires a nonvanishing vacuum expectation value (VEV)

$$
\langle\phi^0\rangle = \sqrt{NL}M_s, \qquad (21)
$$

and we define subtracted fields. In Ref.[10], it is argued that the nonlocal composite field ϕ^0 works as an order parameter for the extended chiral symmetry.

The effective action of the pions and gauge boson is obtained by a similar method as before. For detailes, see Ref.[11]. Especially the pion mass is obtained at *finite lattice spacing* as follows,

$$\epsilon \propto \frac{M_B a^2}{M_s}. \tag{22}$$

This expression (22) should be compared with (13). In the second lattice model with the exact extended chiral symmetry, the Goldstone theorem holds and the (pion mass)$^2 \propto M_B$ at finite lattice spacing. This result is welcome but it should be remarked that this result is obtained in the weak-coupling (or large N) region. We also verified that the PCAC relation appears in the desired form at finite lattice spacing.

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